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AN INTRODUCTION TO MATHEMATICAL COSMOLOGY

This book provides a concise introduction to the mathematical aspects of the origin, structure and evolution of the universe. The book begins with a brief overview of observational and theoretical cosmology, along with a short introduction to general relativity. It then goes on to discuss Friedmann models, the Hubble constant and deceleration parameter, singularities, the early universe, inflation, quantum cosmology and the distant future of the universe. This new edition contains a rigorous derivation of the Robertson–Walker metric. It also discusses the limits to the parameter space through various theoretical and observational constraints, and presents a new inflationary solution for a sixth degree potential.

This book is suitable as a textbook for advanced undergraduates and beginning graduate students. It will also be of interest to cosmologists, astrophysicists, applied mathematicians and mathematical physicists.

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AN INTRODUCTION TO MATHEMATICAL COSMOLOGY

Second edition

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