Perspectives in Mathematical Logic

Ω-Group:
R. O. Gandy, H. Hermes, A. Levy, G. H. Müller,
G. E. Sacks, D. S. Scott
Petr Hájek
Pavel Pudlák

Metamathematics of First-Order Arithmetic

Springer
Dedicated to our wives, Marie and Vera
Preface to the Series
Perspectives in Mathematical Logic
(Edited by the "Ω-group for Mathematical Logic" of the Heidelberger Akademie der Wissenschaften)

On Perspectives. Mathematical logic arouse from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly in the last two decades, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps of guides to this complex terrain. We shall not aim at encyclopaedic coverage: nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain themes, it will be of greater interest than a mere assemblage of results and techniques.

The books in the series differ in level: some are introductory, some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their books with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of values, the credit will be theirs.

History of the Ω-Group. During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R.O. Gandy, A. Levy, G.H. Müller, G. Sacks, D.S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F.K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans
Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and that of the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the over all plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors’ ideas and our own. Often the battle ended with a detailed plan for a better book which the author was keen to write and which would indeed contribute a perspective.

Oberwolfach, September 1975

Acknowledgements. In starting our enterprise we essentially were relying on the personal confidence and understanding of Professor Martin Barner of the Mathematisches Forschungsinstitut Oberwolfach, Dr. Klaus Peters of Springer-Verlag and Dipl.-Ing. Penschuck of the Stiftung Volkswagenwerk. Through the Stiftung Volkswagenwerk we received a generous grant (1970–1973) as an initial help which made our existence as a working group possible.

Since 1974 the Heidelberger Akademie der Wissenschaften (Mathematisch-Naturwissenschaftliche Klasse) has incorporated our enterprise into its general scientific program. The initiative for this step was taken by the late Professor F.K. Schmidt, and the former President of the Academy, Professor W. Doerr.

Through all the years, the Academy has supported our research project, especially our meetings and the continuous work on the Logic Bibliography, in an outstandingly generous way. We could always rely on their readiness to provide help wherever it was needed.

Assistance in many various respects was provided by Drs. U. Felgner and K. Gloede (till 1975) and Drs. D. Schmidt and H. Zeitler (till 1979). Last but not least, our indefatigable secretary Elfriede Ihrig was and is essential in running our enterprise.

We thank all those concerned.

Heidelberg, September 1982

R. O. Gandy
A. Levy
G. Sacks

H. Hermes
G. H. Müller
D. S. Scott
Authors' Preface

After having finished this book on the metamathematics of first order arithmetic, we consider the following aspects of it important: first, we pay much attention to subsystems (fragments) of the usual axiomatic system of first order arithmetic (called Peano arithmetic), including weak subsystems, i.e. so-called bounded arithmetic and related theories. Second, before discussing proper metamathematical questions (such as incompleteness) we pay considerable attention to positive results, i.e. we try to develop naturally important parts of mathematics (notably, some parts of set theory, logic and combinatorics) in suitable fragments. Third, we investigate two notions of relative strength of theories: interpretability and partial conservativity. Fourth, we offer a systematic presentation of relations of bounded arithmetic to problems of computational complexity.

The need for a monograph on metamathematics of first order arithmetic has been felt for a long time; at present, besides our book, at least two books on this topic are to be published, one written by R. Kaye and one written by C. Smoryński. We have been in contacts with both authors and are happy that the overlaps are reasonably small so that the books will complement each other.

This book consists of a section of preliminaries and of three parts: A – Positive results on fragments, B – Incompleteness, C – Bounded arithmetic. Preliminaries and parts A, B were written by P. H., part C by P. P. We have tried to keep all parts completely compatible.

The reader is assumed to be familiar with fundamentals of mathematical logic, including the completeness theorem and Herbrand’s theorem; we survey the things assumed to be known in the Preliminaries, in order to fix notation and terminology.

Acknowledgements. Our first thanks go to the members of the Ω-group for the possibility of publishing the book in the series Perspectives in mathematical logic and especially to Professor Gert H. Müller, who invited P. H. to write a monograph with the present title, agreed with his wish to write the book jointly with P. P. and continuously offered every possible help. We
are happy to recognize that we have been deeply influenced by Professor Jeff Paris. Soon after the famous independence results of Paris, Kirby and Harrington, Jeff Paris repeatedly visited Prague and gave talks about the research of his Manchester group. Since then, he has come to Prague many times and we always learn much from him. On various occasions we met other mathematicians working in this field (Adamowicz, Buss, Clote, Dimitracopoulos, Feferman, Kaye, Kossak, Kotlarski, Lindström, Montagna, Ressayre, Simpson, Smoryński, Solovay, Takeuti, Wilkie, Woods and others) and many of them visited Czechoslovakia. Discussions with them and preprints of their papers have been an invaluable source of information for us. We have profited extremely much from our colleagues J. Krajíček and V. Švejdar and other members of our Prague seminar. The Mathematical Institute of the Czechoslovak Academy of Sciences has been a good working place. Several people have read parts of the manuscript and suggested important improvements. Our thanks especially to Peter Clote, William Eldridge, Richard Kaye, Juraj Hromkovič and Jiří Sgall for their help. Mrs. K. Trojanová and Mrs. D. Berková helped us considerably with typing; and D. Harmanec provided valuable technical help with the preparation of the bibliography on a computer. Last but not least, our families have got used to sacrifice for our scientific work. They deserve our most cordial thanks.

November 1990

Petr Hájek

Pavel Pudlák
# Table of Contents

## Preliminaries
(1) Some Logic .................................................. 5
(2) The Language of Arithmetic, the Standard Model ........ 12
(3) Beginning Arithmetization of Metamathematics ......... 20

## PART A

### CHAPTER I

Arithmetic as Number Theory, Set Theory and Logic .......... 27
Introduction ..................................................... 27

1. Basic Developments; Partial Truth Definitions .......... 28
   (a) Properties of Addition and Multiplication,
       Divisibility and Primes .................................. 28
   (b) Coding Finite Sets and Sequences; the Theory \( I\Sigma_0(\text{exp}) \) ..... 37
   (c) Provably Recursive Functions; the Theory \( I\Sigma_1 \) .......... 44
   (d) Arithmetization of Metamathematics: Partial Truth Definitions 50

2. Fragments of First-Order Arithmetic ....................... 61
   (a) Induction and Collection ................................ 61
   (b) Further Principles and Facts About Fragments .......... 67
   (c) Finite Axiomatizability; Partial Truth Definitions
       for Relativized Arithmetical Formulas ................. 77
   (d) Relativized Hierarchy in Fragments ..................... 81
   (e) Axiomatic Systems of Arithmetic with No Function Symbols 86

3. Fragments and Recursion Theory ............................ 89
   (a) Limit Theorem ........................................... 89
   (b) Low Basis Theorem ...................................... 91
   (c) Infinite \( \Delta_1 \) Subsets ............................... 95
   (d) Matiyasevič’s Theorem in \( I\Sigma_1 \) .................. 97

4. Elements of Logic in Fragments .............................. 98
   (a) Arithmetizing Provability ................................ 98
Chapter II

Fragments and Combinatorics

1. Ramsey's Theorems and Fragments
   (a) Statement of Results
   (b) Proofs (of 1.5, 1.7, 1.9)
   (c) Proofs (of 1.6, 1.8, 1.10)

2. Instances of the Paris-Harrington Principle and Consistency Statements
   (a) Introduction and Statement of Results
   (b) Some Combinatorics
   (c) Proof of Con*(IΣn + Tr(PH)) → (PH)u (for u ≥ 1)
   (d) Strong Indiscernibles
   (e) Final Considerations

3. Schwichtenberg-Wainer Hierarchy and α-large Sets
   (a) Ordinals in IΣ1
   (b) Transfinite Induction and Fragments
   (c) α-large Sets in IΣ1
   (d) Schwichtenberg-Wainer Hierarchy

Part B

Chapter III

Self-Reference

1. Preliminaries
   (a) Interpretability and Partial Conservativity
   (b) Theories Containing Arithmetic; Sequential Theories; PA and ACA0
   (c) Numerations and Binumerations

2. Self-Reference and Gödel's Theorems, Reflexive Theories
   (a) Existence of Fixed Points
   (b) Gödel's First Incompleteness Theorem and Related Topics
   (c) Gödel's Second Incompleteness Theorem
   (d) Pure Extensions of PA
   (e) Interpretability in Pure Extensions of PA

3. Definable Cuts
   (a) Definable Cuts and Their Properties
   (b) A Strong Form of Gödel's Second Incompleteness Theorem
   (c) Herbrand Provability and Herbrand Consistency
   (d) Cuts and Interpretations

4. Partial Conservativity and Interpretability
   (a) Some Prominent Examples
(b) General Theorems on Partial Conservativity;
   Some Fixed-Point Theorems .................................. 195
(c) Applications, Mainly to Interpretability .................. 206

CHAPTER IV
Models of Fragments of Arithmetic .............................. 213
1. Some Basic Constructions .................................... 214
   (a) Preliminaries ........................................... 214
   (b) Definable Ultrapower of the Standard Model .............. 216
   (c) On Submodels and Cuts .................................. 218
   (d) Models for the Hierarchy ................................ 220
   (e) Elementary End Extensions ............................... 227
   (f) A Conservation Result ................................... 230
2. Cuts in Models of Arithmetic with a Top ...................... 232
   (a) Arithmetic with a Top and Its Models .................... 232
   (b) Cuts ................................................... 234
   (c) Extendable, Restrainable and Ramsey Cuts ............... 236
   (d) Satisfaction in Finite Structures with an Application
to Models of $I\Sigma_1$ .................................... 241
3. Provably Recursive Functions and the Method of Indicators .... 245
   (a) Provably Recursive Functions, Envelopes ................. 245
   (b) Indicators and Paris Sequences ........................... 247
   (c) Paris Sequences of the First Kind ....................... 250
   (d) Paris Sequences of the Second Kind ...................... 253
   (e) Further Consequences .................................. 257
4. Formalizing Model Theory .................................... 258
   (a) Some Results on Satisfaction and Consistency ......... 259
   (b) A Conservation Result in $I\Sigma_1$ .................... 260
   (c) Appendix: Another Conservation Result .................. 263

PART C
CHAPTER V
Bounded Arithmetic ............................................. 267
1. A Survey of Weak Fragments of Arithmetic .................. 268
   (a) Fragments of Arithmetic ................................ 268
2. A Brief Introduction to Complexity Theory .................. 276
   (a) Time and Space Complexity Classes ..................... 277
   (b) Nondeterministic Computations ........................... 279
   (c) Degrees and $NP$-completeness ........................... 280
   (d) Oracle Computations .................................... 282
   (e) The Linear Time Hierarchy and the Polynomial Hierarchy 283
   (f) Nepomničič’s Theorem ................................... 285
   (g) The Diagonal Method for Separating Complexity Classes 288
3. Exponentiation, Coding Sequences
   and Formalization of Syntax in \( I\Sigma_0 \) .............................................. 294
   (a) Introduction ................................................................................. 294
   (b) Sets and Sequences ..................................................................... 295
   (c) The Exponentiation Relation ..................................................... 299
   (d) Developing \( I\Sigma_0 + \Omega_1 \) ...................................................... 303
   (e) The Number of Ones in a Binary Expansion ............................... 304
   (f) Coding Sequences ...................................................................... 309
   (g) Syntactical Concepts .................................................................. 312
   (h) Formalizations Based on Context-Free Grammars ....................... 315

4. Witnessing Functions ..................................................................... 320
   (a) Introduction ............................................................................... 320
   (b) Fragments of Bounded Arithmetic ............................................. 320
   (c) Definability of Turing Machine Computations 
      in Fragments of Bounded Arithmetic ............................................. 330
   (d) Witnessing Functions .................................................................. 337
   (e) On the Finite Axiomatizability of Bounded Arithmetic ............... 350

5. Interpretability and Consistency .................................................... 360
   (a) Introduction ............................................................................... 360
   (b) Truth Definitions for Bounded Formulae ..................................... 361
   (c) An Interpretation of \( I\Sigma_0 \) in \( Q \) ............................................. 366
   (d) Cut-Elimination and Herbrand's Theorem 
      in Bounded Arithmetic .................................................................. 371
   (e) The \( \Pi_1 \) Theorems of \( I\Sigma_0 + \text{Exp} \) .................................. 380
   (f) Incompleteness Theorems ......................................................... 386
   (g) On the Limited Use of Exponentiation ....................................... 393

Bibliographical Remarks and Further Reading .................................. 397

Bibliography ..................................................................................... 409

Index of Terms .................................................................................. 455

Index of Symbols .............................................................................. 459
People have been interested in natural numbers since forever. The ancient mathematicians knew and used the principle of descente infinie, which is a form of mathematical induction. The principle is as follows: if you want to show that no number has the property \( \varphi \), it suffices to show that for each number \( n \) having the property \( \varphi \) there is a smaller number \( m < n \) having the property \( \varphi \). (If there were a number having \( \varphi \) we could endlessly find smaller and smaller numbers having \( \varphi \), which is absurd.) The Greeks used the principle for a proof of incommensurability of segments. The principle was rediscovered in modern times by P. Fermat (1601–1665). The principle of mathematical induction itself (if 0 has the property \( \varphi \) and for each number \( n \) having \( \varphi \) also \( n + 1 \) has \( \varphi \) then all numbers have \( \varphi \)) seems to have been first used by B. Pascal (1623–1662) in a proof concerning his triangle. A general formulation appears in a work of J. Bernoulli (1654–1705). (Our source is [Meschkowski 78–81].)

In 1861 Grassman published his Lehrbuch der Arithmetik; in our terms, he defines integers as an ordered integrity domain in which each non-empty set of positive elements has a least element. In 1884 Frege's book Grundlagen der Arithmetik was published. We can say that Frege's natural numbers are classes; each such class consists of all sets of a certain fixed finite cardinality. (Frege speaks of concepts, not of classes.) The famous Dedekind's work Was sind und was sollen die Zahlen appears in 1888. Dedekind's natural numbers are defined as a set \( N \) together with an element \( 1 \in N \) a one-one mapping \( f \) of \( N \) into itself such that \( 1 \) is not in the range of \( f \) and \( N \) is the smallest set containing 1 and closed under \( f \). Dedekind and Frege agreed that arithmetic is a part of logic, but differed in their opinions on what logic is. They both used the same main device: a one-one mapping and closedness under that mapping.

Dedekind was not interested in finding a formal deductive system for natural numbers; this was the main aim of Peano's investigation of natural numbers (Arithmetices principia nova methoda exposita, 1889). Peano's axiom system (taken over from Dedekind, who had it from Grassman) is, in our terminology, second order: it deals with numbers and sets of numbers. Nowadays